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Inverse Spectral Theory for Random Jacobi Matrices

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We give necessary and sufficient conditions for a Herglotz function to be the w-function of a random stationary Jacobi matrix.

KEY WORDS: Inverse spectral theory; random Jacobi matrices.

1. INTRODUCTION

There is undoubtedly much to gain from a clear understanding of Mark Kac's deep contributions to mathematical physics and probability theory. We would like to pay homage to his joint work with P. van Moerbeke on the inverse spectral theory of periodic Jacobi matrices and its application to the problem of the Toda lattice.[5,6] We propose an extension of the inverse spectral theory of Jacobi matrices to the random stationary case. We hope to be able to discuss the problems of some "random Toda Lattices" in the near future.

Random Schrödinger operators appearing in the mathematical theory of quantum disordered media and their spectral properties have been under very active investigation during the last decade; for review see, for example, Refs. 1, 2, 15, or Chapter 9 of Ref. 13. Since the assumptions of stationarity and ergodicity hold most of the time, they appear as natural generalizations of periodic and almost periodic Schrödinger operators. This last point is important in understanding the effect of randomness on the spectral properties. In the periodic case, both direct and inverse spectral theories have been known for a long time. In the almost periodic case, both

This paper is dedicated to the memory of Mark Kac.

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theories are recent and still incomplete. In the random case, a good account of the direct theory can be found in the reviews quoted above. The first attempt to attack the inverse problems is due to Kotani.[8] He recently gave a more complete version of his theory.[9] His results deal with continuous, one-dimensional Schrödinger operators. Their discrete analogs are infinite Jacobi matrices. We propose to extend Kotani's work to this case. As noticed in Refs. 12 and 7, there are important differences between the continuous and the lattice cases. Nevertheless, the strategy is the same: summarize, as in the periodic case, most of the spectral properties in the average of some functions (such as the average of the Green's function or the Weyl–Titchmarsch m-functions), find properties of these functions, and try to reconstruct random operators from these functions. Obviously, analytic and Herglotz functions are bound to play a crucial role. Also, the inverse spectral theory for periodic operators is of prime importance, for the reconstruction in the general random case is based on an approximation argument by periodic analogs.

The terminology and the notations of the deterministic case are introduced in Section 2. Section 3 deals with the definition and the study of the w-function of a stationary Jacobi matrix. We believe that most of the results are essentially known. We present them in a self-contained way. The net result is that such a function w is in the class $\mathscr{H}$ of Herglotz functions on $\mathbb{C}_+$, as is its derivative; its range is contained in $\{z \in \mathbb{C};\ \operatorname{Im} z \in [0, \pi],\ \operatorname{Re} z \leqslant c\}$ for some $c \in \mathbb{R}$; and $w(\lambda)$ is equivalent to $\log(-1/\lambda)$ as $\lambda \to \infty$. The original part of the paper is contained in Section 4. After some technical lemmas on such Herglotz functions, we show the existence of a stationary Jacobi matrix having for w-function the Herglotz function we started from. The construction relies on the approximation of such Herglotz functions by w-functions of periodic Jacobi matrices as characterized in the work of Kac and van Moerbeke.

2. DETERMINISTIC NOTATIONS AND PRELIMINARIES

Let us assume that $\{a_n;\ n \in \mathbb{Z}\}$ and $\{b_n;\ n \in \mathbb{Z}\}$ are given sequences of real numbers. On the Hilbert space $l^2 = l^2(\mathbb{Z})$ we consider the operator H defined formally by

$$(Hf)_n = a_n f_{n+1} + a_{n-1} f_{n-1} + b_n f_n \tag{2.1}$$

To obtain a natural self-adjoint realization of H, we proceed as follow: let

$$\mathscr{D}_0 = \{f = (f_n)_{n \in \mathbb{Z}} \in l^2;\ f_n = 0 \text{ for all but finitely many } n \in \mathbb{Z}\}$$

$$\mathscr{D}_1 = \{f = (f_n)_{n \in \mathbb{Z}} \in l^2;\ (a_n f_{n+1} + a_{n-1} f_{n-1} + b_n f_n)_n \in l^2\}$$

and let us define the operators $H_{\min}$ with domain $\mathscr{D}_0$ and $H_{\max}$ with domain $\mathscr{D}_1$ by the formula (2.1). One easily checks that $H_{\min}$ and $H_{\max}$ are symmetric and that $(H_{\min})^* = H_{\max}$. Moreover, the condition

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2} = \sum_{n=-1}^{-\infty} \frac{1}{a_n^2} = \infty \tag{2.2}$$

implies that $H_{\min}$ is essentially self-adjoint and that its unique self-adjoint extension is $H_{\max}$. The latter will be denoted by H form now on.

The investigation of the eigenvalue equation

$$a_n f_{n+1} + a_{n-1} f_{n-1} + b_n f_n = \lambda f_n \tag{2.3}$$

will be of crucial importance in the study of the spectral properties of the operator H. We want to use the formalism of transfer matrices and for this reason it is very convenient to assume that

$$a_n \neq 0, \qquad n \in \mathbb{Z}$$

If $f = (f_n)_n$ is any solution of the eigenvalue equation (2.3), we have

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = M^{(n)} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix}$$

with

$$M^{(n)} = M_n(\lambda) \cdots M_1(\lambda)$$

and

$$M_k(\lambda) = \begin{pmatrix} (\lambda - b_k)/a_k & -a_{k-1}/a_k \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

for $k \geqslant 1$. If $g = (g_n)_n$ is another solution, we can write

$$\begin{pmatrix} f_{n+1} & g_{n+1} \\ f_n & g_n \end{pmatrix} = M^{(n)}(\lambda) \begin{pmatrix} f_1 & g_1 \\ f_0 & g_0 \end{pmatrix} \tag{2.5}$$

and this gives

$$a_n[f_{n+1} g_n - f_n g_{n+1}] = a_0[f_1 g_0 - f_0 g_1]$$

by using (2.4) and (2.5) to compute the determinant of $M^{(n)}(\lambda)$. If f and g are any elements of $\mathbb{R}^{\mathbb{Z}}$, we set

$$W(f, g)_n = a_n[f_{n+1} g_n - f_n g_{n+1}], \qquad n \in \mathbb{Z} \tag{2.6}$$

for the Wronskian of f and g. Consequently, we just proved that the Wronskian of two solutions is a constant and one easily sees that this constant is different from zero if and only if the two solutions are linearly independent.

We now introduce another important quantity in the spectral theory of differential and difference equations, the so-called Titchmarsh–Weyl m-function. We follow basically the Appendix of Ref. 7.

As before, we assume that we are given two real sequences $\{a_n; n \in \mathbb{Z}\}$ and $\{b_n; n \in \mathbb{Z}\}$. We define the operator H_+ on $l^2[1, \infty)$ as the restriction of the operator H to $[1, \infty)$ with Dirichlet boundary condition at 0. In fact, we set

$$\mathscr{D}_0^+ = \{\psi = (\psi_n)_{n \geqslant 1} \in l^2[1, \infty); \psi_n = 0 \text{ for } n \text{ large}\}$$

$$\mathscr{D}_1^+ = \left\{\psi = (\psi_n)_{n \geqslant 1} \in l^2[1, \infty); \sum_{n \geqslant 2} |a_n \psi_{n+1} + a_{n-1}\psi_{n-1} + b_n \psi_n|^2 < +\infty\right\}$$

and

$$(H_+ \psi)_n = \begin{cases} a_1 \psi_2 + b_1 \psi_1 & \text{if} \quad n = 1 \\ a_n \psi_{n+1} + a_{n-1} \psi_{n-1} + b_n \psi_n & \text{if} \quad n \geqslant 2 \end{cases} \tag{2.7}$$

for $\psi \in \mathscr{D}_1^+$, and we denote by $H_{+,\min}$ the operator H_+ with domain $\mathscr{D}_0^+$ and by $H_{+,\max}$ the operator H_+ with domain $\mathscr{D}_1^+$. It is easy to check that $H_{+,\max}$ is symmetric and that $(H_{+,\min})^* = H_{+,\min}$. Moreover, the condition:

$$\sum_{n \geqslant 1} \frac{1}{a_n^2} = \infty \tag{2.2$_+$}$$

implies that $H_{+,\min}$ is essentially self-adjoint and its unique self-adjoint extension is $H_{+,\max}$. From now on we will denote this unique self-adjoint extension by H_+. For $\lambda \in \mathbb{C}_+$, we let $f_{\lambda,0}$ be the solution of the eigenvalue equation (2.3) that satisfies

$$f_{\lambda,0}(0) = 0, \qquad f_{\lambda,0}(1) = 1$$

and we denote by $f_{\lambda,\infty}$ any solution that is square summable at $+\infty$. Note that the space of such solutions is one-dimensional. Let $G_\lambda^+(m, n)$ be the symmetric kernel defined by:

$$G_\lambda^+(m, n) = -\frac{1}{W(f_{\lambda,0}, f_{\lambda,\infty})} f_{\lambda,0}(m) f_{\lambda,\infty}(n)$$

when $m \leqslant n$. Note also that this definition is independent of the particular choice of $f_{\lambda,\infty}$. Now, G_λ^+ is easily seen to be the kernel of the resolvent operator $(H_+ - \lambda)^{-1}$. Consequently, if one defines the Titchmarsh–Weyl function m by

$$m_+(\lambda) = \langle (H_+ - \lambda)^{-1} \delta_1, \delta_1 \rangle \tag{2.8}$$

one has

$$m_+(\lambda) = G_\lambda^+(1, 1) = -\frac{1}{a_0} \frac{f_{\lambda,\infty}(1)}{f_{\lambda,\infty}(0)} \tag{2.9}$$

For $\operatorname{Im} \lambda > 0$ we have

$$|m_+(\lambda)| \leqslant \|(H_+ - \lambda)^{-1}\| \leqslant 1/\operatorname{Im} \lambda \tag{2.10}$$

and, using the first resolvent equation,

$$\operatorname{Im} m_+(\lambda) = \operatorname{Im} \lambda \, \|(H_+ - \lambda)^{-1} \, \delta_1\|^2 \geqslant \frac{\operatorname{Im} \lambda}{|b_1 - \lambda|^2 + a_1^2} \tag{2.11}$$

Now, for each $k = 0, 1, 2,...$, let $\{a_n^{(k)}; n \geqslant 1\}$ be a sequence of real numbers satisfying $(2.2)_+$, $\{b_n^{(k)}; n \geqslant 1\}$ be any sequence of real numbers, and let us denote by $H_+^{(k)}$ the corresponding operator. Then, if

$$\lim_{k \to \infty} a_n^{(k)} = a_n^{(0)}$$

and

$$\lim_{k \to \infty} b_n^{(k)} = b_n^{(0)}$$

for all $n = 1, 2,...$, one has that $H_+^{(k)}$ converges to $H_+^{(0)}$ in the strong resolvent sense because $\mathscr{D}_0^+$ is a common core and $H_+^{(k)} f$ converges to $H_+^{(0)} f$ for all f in $\mathscr{D}_0^+$. Using the definition (2.8) of the m-function, one obtains that

$$\lim_{k \to \infty} m_+^{(k)}(\lambda) = m_+^{(0)}(\lambda) \tag{2.12}$$

uniformly in compact subsets of $\mathbb{C}_+$.

Note also that the m-function $m_+(\lambda)$ belongs to the Herglotz class $\mathscr{H}$. The latter is the class of holomorphic functions in the upper half-plane $\mathbb{C}_+$ with positive imaginary part. Its relevance to the spectral theory of differential and difference operators has been known for a long time and is particularly well emphasized in Ref. 9.

We will need the following technical result.

Lemma 2.1. Let us assume that the sequence $\{(a_n, b_n); n \geqslant 1\}$ is bounded. Then the function $\lambda \to \log m_+(\lambda) - \log(-1/\lambda)$ is holomorphic at $\lambda = \infty$ and has the Taylor expansion

$$\log m_+(\lambda) - \log \frac{1}{-\lambda} = \frac{1}{\lambda} s_1 + \frac{1}{\lambda^2}\left(s_2 - \frac{1}{2} s_1^2\right) + \frac{1}{\lambda^3}\left(s_3 - s_1 s_2 + \frac{1}{3} s_1^3\right)$$
$$+ \frac{1}{\lambda^4}\left(s_4 - \frac{1}{2} s_2^2 - s_1 s_3 + s_1 s_2^2 - \frac{1}{4} s_1^4\right) + \cdots \quad (2.13)$$

where $s_n = \langle H_+^n \delta_1, \delta_1 \rangle$, $n = 1, 2,\ldots$. The above series converges for $|\lambda| \geqslant 2 \|H_+\|$.

Proof. We first notice that

$$\log m_+(\lambda) = \log\langle (H_+ - \lambda)^{-1} \delta_1, \delta_1 \rangle$$
$$= \log \frac{1}{-\lambda} + \log \left\langle \left(I - \frac{1}{\lambda} H_+\right)^{-1} \delta_1, \delta_1 \right\rangle$$

Now, the function

$$f(z) = \log\langle (I - zH_+)^{-1} \delta_1, \delta_1 \rangle$$

is holomorphic on the disk $|z| < \frac{1}{2} \|H_+\|^{-1}$ and

$$f(z) = \log\left(1 + \sum_{n=1}^{\infty} s_n z^n\right)$$
$$= \sum_{n=1}^{\infty} s_n z^n - \frac{1}{2}\left(\sum_{n=1}^{\infty} s_n z^n\right)^2 + \frac{1}{3}\left(\sum_{n=1}^{\infty} s_n z^n\right)^3 + \cdots$$
$$= s_1 z + \left(s_2 - \frac{1}{2} s_1^2\right) z^2 + \left(s_3 - s_1 s_2 + \frac{1}{3} s_1^3\right) z^3 + \cdots$$

which gives (2.13) by taking $z = 1/\lambda$. ▮

3. RANDOM CASE: DIRECT THEORY

We first describe the setting of the present section.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space and let T be a bimeasurable invertible transformation of Ω which preserves $\mathbf{P}$. On such a probability space we consider a stationary sequence $\{(a_n, b_n); n \in \mathbb{Z}\}$ of random variables such that:

(i) The common distribution of the a_n, say μ_a, is concentrated on $(0, \infty)$ and satisfies

$$\int_{(0,\infty)} (x^2 + |\log x|)\, \mu_a(dz) < \infty \tag{3.1}$$

(ii) The common distribution of the b_n, say μ_b, is concentrated on $\mathbb{R}$ and satisfies:

$$\int_{\mathbb{R}} x^2 \mu_b(dx) < +\infty \tag{3.2}$$

The above assumption will be in force throughout this section.

We first note that

$$\mathbf{P}\left\{\sum_{n=1}^{\infty} \frac{1}{a_n^2} = \infty\right\} = 1$$

because, if this was not the case, we would have

$$\mathbf{P}\left\{\sum_{n=1}^{\infty} \frac{1}{a_n^2} < \infty,\, a_1 < M\right\} > 0$$

for $M > 0$ large enough, and Poincaré's recurrence lemma would imply that

$$\mathbf{P}\left\{\sum_{n=m}^{\infty} \frac{1}{a_n^2} < +\infty,\, a_m < M \text{ for infinitely many } m\right\} > 0$$

which is impossible. Consequently, for $\mathbf{P}$-almost all $\omega \in \Omega$, one can define a self-adjoint operator $H_+(\omega)$ on $\mathscr{D}_1^+$ by formula (22.7) and this operator is essentially self-adjoint on $\mathscr{D}_0^+$. One defines similarly self-adjoint operators $H_-(\omega)$ and $H(\omega)$ on $\mathscr{D}_1^-$ and $\mathscr{D}_1$, respectively, and they are essentially self-adjoint on $\mathscr{D}_0^-$ and $\mathscr{D}_0$, respectively.

Consequently, we can define

$$m_+(\lambda, \omega) = \langle [H_+(\omega) - \lambda]^{-1} \delta_1, \delta_1 \rangle \tag{3.3}$$

for $\operatorname{Im} \lambda > 0$ and we have

$$m_+(\lambda, \omega) = -\frac{1}{a_0(\omega)} \frac{f_{\lambda,\infty}(1, \omega)}{f_{\lambda,\infty}(0, \omega)} \tag{3.4}$$

according to (2.9). This implies that

$$m_+(\lambda, T^n\omega) = -\frac{1}{a_n(\omega)} \frac{f_{\lambda,\infty}(n+1, \omega)}{f_{\lambda,\infty}(n, \omega)}$$

and

$$\frac{1}{n+1} \log |f_{\lambda,\infty}(n+1, \omega)| = \frac{1}{n+1} \log |f_{\lambda,\infty}(0, \omega)| + \frac{1}{n+1} \sum_{k=0}^{n} \log |a_0 m_+(\lambda)| \circ T^k \omega$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log |f_{\lambda,\infty}(n, \omega)| = \mathbf{E}\{\log a_0\} + \mathbf{E}\{\log |m_+(\lambda)|\} \tag{3.5}$$

whenever the probability measure $\mathbf{P}$ is ergodic for the shift T. In this case one can also define the *integrated density of states* $d\mathbf{n}(\xi)$ as the "nonrandom" probability measure that appears as the almost sure limit as $n \to \infty$ of the "random" probability measures $d\mathbf{n}_n(\xi)$, where $\mathbf{n}_n(\xi)$ is the average number of eigenvalues of the restriction of $d\mathbf{n}_n(\xi)$ to $\{1,..., n\}$, with Dirichlet boundary condition at $n+1$, which are not greater than ξ. The existence of the integrated density of states in the ergodic case is well known (see, for example, Refs. 1, 2, or Chapter 9 of Ref. 13).

In the present general situation in which we do not want to assume the ergodicity of the sequence $\{(a_n, b_n); n \in \mathbb{Z}\}$, one can still define an integrated density of state measure $d\mathbf{n}(\xi)$ by defining the nondecreasing function $n(\xi)$ as the limit of the expectations of the random nondecreasing functions $\mathbf{n}_n(\xi)$ described above. The fact that this limit exists can easily be seen from the proof given in Chapter V of Ref. 2. Note that $d\mathbf{n}(\xi)$ has total mass 1, so it is a probability measure on $\mathbb{R}$.

One of the main attractions of the integrated density of states is the so-called Thouless formula, which relates it to the *Liapunov exponent* $\gamma(\lambda)$ defined by

$$\gamma(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log \|M^{(n)}(\lambda, \omega)\| \tag{3.6}$$

in the ergodic case. The existence of this almost sure limit is guaranteed by the subadditive ergodic theorem. In the present situation the Lyapunov exponent $\gamma(\lambda)$ is defined by

$$\gamma(\lambda) = \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\{\log \|M^{(n)}(\lambda, \omega)\|\} \tag{3.6$'$}$$

The limit in the right-hand side exists because the sequence is subadditive. Moreover, one easily sees that it is nonnegative. In the ergodic case, the Liapunov exponent governs the almost sure exponential behavior of the transfer matrices. In general we will have only the weaker form (3.6)′. Nevertheless, the proof of the Thouless formula (see, for example, Section 2.6 of Ref. 1) can be pplied to the present situation by taking expectations at an appropriate step of the proof. This formula reads

$$\gamma(\lambda) = \int \log |\xi - \lambda| \, d\mathbf{n}(\xi) - \mathbf{E}\{\log a_0\} \tag{3.7}$$

Using the bounds (2.10) and (2.11), one immediately sees that the quantity

$$w(\lambda) = \mathbf{E}\{\log m_+(\lambda)\} \tag{3.8}$$

makes sense for all $\lambda \in \mathbb{C}_+$ because our assumptions (i) and (ii) imply that

$$\mathbf{E}\{\log(1 + |b_1| + |a_1|)\} < +\infty$$

Obviously, the function w defined by (3.8) is in the Herglotz class. Moreover, we have the following result:

Proposition 3.1. Under conditions (3.1) and (3.2), there exists a probability measure on $\mathbb{R}$, say $\mathbf{n}$, such that

$$\int \xi \, d\mathbf{n}(\xi) = \mathbf{E}\{b_0\} \tag{3.9}$$

$$\int \xi^2 \, d\mathbf{n}(\xi) \leqslant 2\mathbf{E}\{2a_0^2 + b_0^2\} \tag{3.10}$$

and such that

$$w(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}(\xi) \tag{3.11}$$

and

$$\operatorname{Re} w(\lambda) \leqslant \mathbf{E}\{\log(1/a_0)\} < +\infty \tag{3.12}$$

for $\lambda \in \mathbb{C}_+$.

Proof. The existence of the probability measure $\mathbf{n}$ has been argued in our discussion of the integrated density of states. The Thouless formula and the fact that

$$-\gamma(\lambda) - \mathbf{E}\{\log a_0\} = \operatorname{Re} w(\lambda) \tag{3.13}$$

[see (3.5)] give the integrability of the function $\log(1 + |\xi|)$ with respect to $d\mathbf{n}(\xi)$ and

$$-\operatorname{Re} w(\lambda) = \int \log |\xi - \lambda| \, d\mathbf{n}(\xi)$$

and hence we have

$$w(\lambda)=\alpha+\int \log\frac{1}{\xi-\lambda}\,d\mathbf{n}(\xi) \tag{3.14}$$

for some constant α. Note that (3.12) follows from (3.13) and the fact that the Lyapunov exponent is nonnegative. Let us first assume that the distribution of (a_0, b_0) has a bounded support. Then, Lemma 2.1 implies that

$$\begin{aligned} w(\lambda) &= \log\frac{1}{-\lambda}+\frac{1}{\lambda}\mathbf{E}\{s_1\}+\frac{1}{\lambda^2}\mathbf{E}\left\{s_2-\frac{1}{2}s_1^2\right\}+\frac{1}{\lambda^3}\mathbf{E}\left\{s_3-s_1s_2+\frac{1}{3}s_1^3\right\} \\ &\quad +\frac{1}{\lambda^4}E\left\{s_4-\frac{1}{2}s_2^2-s_1s_3+s_1^2s_2-\frac{1}{4}s_1^4\right\}+\cdots \\ &= \log\frac{1}{-\lambda}+\frac{1}{\lambda}\mathbf{E}\{b_0\}+\frac{1}{\lambda^2}\mathbf{E}\left\{a_0^2+\frac{1}{2}b_0^2\right\}+\frac{1}{\lambda^3}\mathbf{E}\left\{a_0^2b_0+\frac{1}{3}b_0^3+a_0^2b_1\right\} \end{aligned} \tag{3.15}$$

The probability measure $\mathbf{n}$ has compact support, say K, because the operators $H(\omega)$ are uniformly bounded. Consequently, coming back to (3.14), one gets

$$\begin{aligned} w(\lambda) &= \alpha+\int_K \log\frac{1}{-\lambda}\left(1-\frac{\xi}{\lambda}\right)^{-1} d\mathbf{n}(\xi) \\ &= \alpha+\log\frac{1}{-\lambda}+\sum_{k=1}^{\infty}\frac{1}{k\lambda^k}\int \xi^k\,d\mathbf{n}(\xi) \end{aligned}$$

Identifying with (3.15), one obtains

$$\alpha=0$$

$$\int \xi\,d\mathbf{n}(\xi)=\mathbf{E}\{b_0\}$$

$$\int \xi^2\,d\mathbf{n}(\xi)=\mathbf{E}\{2a_0^2+b_0^2\}$$

$$\int \xi^3\,d\mathbf{n}(\xi)=\mathbf{E}\{3a_0^2b_0+b_0^3+3a_0^2b_1\}$$

$$\vdots$$

In the general case where we do not assume that $\{a_n, b_n); n \in \mathbb{Z}\}$ is uniformly bounded but simply that $\mathbf{E}\{|\log a_0|\} < +\infty$ and $\mathbf{E}\{a_0^2 + b_0^2\} < +\infty$, we use a truncation procedure to reduce the problem to the previous case. More precisely, for each integer $k \geqslant 1$ we set

$$a_n^{(k)} = a_n \wedge k$$

and

$$b_n^{(k)} = (b_n \wedge k) \vee (-k)$$

Then, according to (2.12), we have

$$\lim_{k \to \infty} w^{(k)}(\lambda) = w(\lambda)$$

but since for each $k \geqslant 1$ we have

$$\int \xi^2 \, d\mathbf{n}^{(k)}(\xi) = \mathbf{E}\{2a_0^{(k)2} + b_0^{(k)2}\} \leqslant \mathbf{E}\{2a_0^2 + b_0^2\}$$

because of the first half of the proof, we conclude easily that

$$w(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}(\xi)$$

and that

$$\int \xi^2 \, d\mathbf{n}(\xi) \leqslant \mathbf{E}\{a2a_0^2 + b_0^2\} \quad \blacksquare$$

We end this section with a short discussion of the function w and its range $w(\mathbb{C}_+)$ in the particular case where the random sequence $\{(a_n, b_n); n \in \mathbb{Z}\}$ is actually *periodic*. Let us denote by N the period and let us review some elements of Floquet's theory (see Ref. 5, 6, or 16 for details). The discriminant

$$\Delta(\lambda) = \operatorname{trace} M^{(N)}(\lambda)$$

is a polynomial of degree N. Let us denote by $\lambda_1^+ < \cdots < \lambda_N^+$ the roots of $\Delta(\lambda) = 2$ and by $\lambda_1^- < \cdots < \lambda_N^-$ the roots of $\Delta(\lambda) = -2$. Then we have

$$\lambda_1^+ < \lambda_1^- \leqslant \lambda_2^- < \lambda_2^+ \leqslant \lambda_3^+ < \cdots$$

and the spectrum of H, say $\Sigma(H)$, is a finite union of closed intervals:

$$\Sigma(H) = [\lambda_1^+, \lambda_1^-] \cup [\lambda_2^-, \lambda_2^+] \cup [\lambda_3^+, \lambda_3^-] \cup \cdots$$

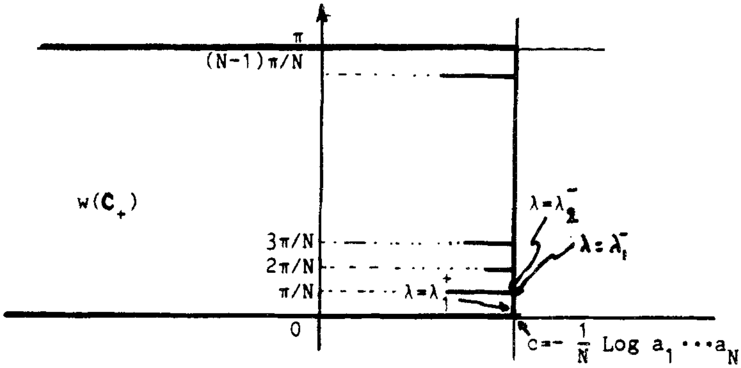

Figure 1

The quantity $\pi n(\lambda) = \operatorname{Im} w(\lambda + i0)$ vanishes when $\lambda \in (-\infty, \lambda_1^+]$, then it increases from 0 to π/N on the first stability interval $[\lambda_1^+, \lambda_1^-]$, remains constant and equal to π/N on the first gap $(\lambda_1^-, \lambda_2^-)$, increases from π/N to $2\pi/N$ on the second stability interval $[\lambda_2^-, \lambda_2^+]$, remains constant and equal to $2\pi/N$ on the second gap $(\lambda_2^+, \lambda_3^+)$, increases,.... On the other hand, the Liapunov exponent $\gamma(\lambda) = \mathbf{E}\{\log(1/a_0)\} - \operatorname{Re} w(\lambda + i0)$ decreases from $+\infty$ to 0 when λ increases from $-\infty$ to λ_1^+, it vanishes on the stability intervals of the spectrum, it is strictly positive in the gaps, and increases from 0 to $+\infty$ when λ increases from $\sup \Sigma(H)$ to $+\infty$. In particular, we always have

$$\operatorname{Re} w(\lambda + i0) \leqslant \mathbf{E}\{\log(1/a_0)\} = -\frac{1}{N} \log a_1 \cdots a_N$$

with equality if λ belongs to the spectrum of H.

4. RANDOM CASE: INVERSE THEORY

According to the results of the previous section, the w-function of a random operator $H(\omega)$ constructed from a stationary sequence $\{(a_n, b_n); n \in \mathbb{Z}\}$ satisfying properties (i) and (ii) of Section 3 is such that:

(i) $\operatorname{Im} w \in [0, \pi]$ and $\operatorname{Re} w \in [-\infty, c]$ for some finite constant c.

(ii) $w(\lambda)$ has an expansion of the form

$$w(\lambda) = \log \frac{1}{-\lambda} + \frac{\alpha_1}{\lambda} + \frac{\alpha_2}{\lambda^2} + O(\lambda^{-2}) \tag{4.1}$$

in a neighborhood of $\lambda = \infty$ for some real numbers α_1 and α_2.

(iii) $w' \in \mathscr{H}$.

The purpose of this section is, given a function w satisfying these properties, to construct a stationary sequence $\{(a_n, b_n);\ n \in \mathbb{Z}\}$ satisfying properties (i) and (ii) of Section 3 and such that the corresponding w-function is the one we started from. First we will need a deeper knowledge of the functions of the Herglotz class that can appear in this way.

Recall that the w-function of a random Jacobi matrix of the type we investigated in Section 3 has a representation

$$w(\lambda) = \int \log \frac{1}{\xi - \lambda}\, d\mathbf{n}(\xi) \tag{4.2}$$

in terms of the integrated density of states. The latter is a probability measure on $\mathbb{R}$ that satisfies

$$\int \log(1 + |\xi|)\, d\mathbf{n}(\xi) < +\infty \tag{4.3}$$

We now give a characterization of these Herglotz functions.

Proposition 4.1. Let $w \in \mathscr{H}$ be such that $w' \in \mathscr{H}$ and

$$w(iy) = \log \frac{1}{-iy} + O(1) \tag{4.4}$$

as $y \to +\infty$. Then w admits the representation (4.2) for some probability measure $\mathbf{n}$ on $\mathbb{R}$ that satisfies (4.3).

Proof. Since $w' \in \mathscr{H}$, the measure $(1/\pi) \operatorname{Im} w'(\xi + i\varepsilon)\, d\xi$ converges vaguely as $\varepsilon \downarrow 0$ to a nonnegative measure, say $\mathbf{n}$, on $\mathbb{R}$ that satisfies

$$\int \frac{1}{1 + \xi^2}\, d\mathbf{n}(\xi) < +\infty$$

(see, for example, Ref. 4 or Section 7 of Ref. 9). Similarly, $w \in \mathscr{H}$ implies that the measure $(1/\pi) \operatorname{Im} w(\xi + i\varepsilon)\, d\xi$ converges vaguely to a nonnegative measure ν on $\mathbb{R}$. (Note that $\mathbf{n}$ has to be the derivative of ν in the sense of distributions.) For this measure ν, we have the usual representation of Herglotz functions:

$$w(\lambda) = \alpha + \beta\lambda + \int \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1 + \xi^2} \right) d\nu(\xi) \tag{4.5}$$

for some $\alpha \in \mathbb{R}$ and some $\beta \leqslant 0$, and consequently

$$= \alpha + \beta\lambda + \int \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1 + \xi^2} \right) \mathbf{n}(\xi)\, d\xi$$

if one writes $\mathbf{n}(\xi) = \mathbf{n}((-\infty, \xi])$ for the distribution function of the measure $\mathbf{n}$. Taking derivatives of both sides and integrating by parts, one obtains

$$w'(\lambda) = \beta + \int \frac{1}{\xi - \lambda}\, d\mathbf{n}(\xi)$$

Taking imaginary parts of both sides of (4.5), one gets

$$\operatorname{Im} w(\lambda) = \beta \operatorname{Im} \lambda + \operatorname{Im} \lambda \int \frac{n(\xi)}{|\xi - \lambda|^2}\, d\xi$$

and

$$\operatorname{Im} w(iy) = \beta y + y \int \frac{n(\xi)}{\xi^2 + y^2}\, d\xi$$

by setting $\lambda = iy$ with $y > 0$, and our assumption (4.4) implies $\beta = 0$. Now

$$\begin{aligned} w(\lambda) - w(i) &= \int \left(\frac{1}{\xi - \lambda} - \frac{1}{\xi - i} \right) n(\xi)\, d\xi \\ &= \int \left(\log \frac{1}{\xi - \lambda} - \log \frac{1}{\xi - i} \right) d\mathbf{n}(\xi) \end{aligned}$$

Taking real parts of both sides and setting $\lambda = yi$ with $y \to +\infty$, one gets

$$\log y + O(1) = \int \log \left(1 + \frac{y^2 - 1}{\xi^2 + 1} \right) d\mathbf{n}(\xi)$$

according to our assumption (4.4). This implies that $\mathbf{n}(\mathbb{R}) = 1$ and (4.3). Consequently, our function $w(\lambda)$ is of the form

$$w(\lambda) = \alpha + \int \log \frac{1}{\xi - \lambda}\, d\mathbf{n}(\xi)$$

and assumption (4.4) gives $\alpha = 0$. ∎

Corollary 4.2. Under the above conditions we must have

$$0 \leqslant \operatorname{Im} w(\lambda) \leqslant \pi$$

The following *comparison result* will play an important role in the sequel.

Proposition 4.3. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be probability measures on $\mathbb{R}$ that satisfy

(a) $\int \xi \, d\mathbf{n}_1(\xi) = \int \xi \, d\mathbf{n}_2(\xi) = 0$

(b) $\int \xi^2 \, d\mathbf{n}_1(\xi) < +\infty$

(c) $\mathbf{n}_2$ has compact support

If the holomorphic functions w_1 and w_2 defined on $\mathbb{C}_+$ by

$$w_1(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}_1(\xi), \qquad w_2(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}_2(\xi)$$

satisfy

$$w_1(\mathbb{C}_+) \subset w_2(\mathbb{C}_+) \tag{4.6}$$

then one has

$$\int \xi^2 \, d\mathbf{n}_1(\xi) \geqslant \int \xi^2 \, d\mathbf{n}_2(\xi) \tag{4.7}$$

The proof relies on the two following technical lemmas.

Lemma 4.4. Let $\mathbf{n}$ be a probability measure on $\mathbb{R}$ satisfying $\int \xi \, d\mathbf{n}(\xi) = 0$ and $\int \xi^2 \, d\mathbf{n}(\xi) < +\infty$ and let w be the holomorphic function defined on $\mathbb{C}_+$ by

$$w(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}(\xi)$$

Then, there exists a probability measure σ on $\mathbb{R}$ satisfying

$$\int \xi \, d\sigma(\xi) = 0, \qquad \int \xi^2 \, d\sigma(\xi) \leqslant \frac{1}{2} \int \xi^2 \, d\mathbf{n}(\xi)$$

and such that the function $u(\lambda) = e^{w(z)}$ admits the representation

$$u(\lambda) = \int \frac{1}{\xi - \lambda} \, d\sigma(\xi) \tag{4.8}$$

Proof. We first assume that $\mathbf{n}$ has compact support. One gets

$$\operatorname{Im} w(\lambda) = \int \operatorname{Arg} \frac{1}{\xi - \lambda} \, d\mathbf{n}(\lambda)$$

so that we must have $0 \leqslant \operatorname{Im} w(\lambda) \leqslant \pi$ for all $\lambda \in \mathbb{C}_+$. This implies that u is in the Herglotz class $\mathscr{H}$ and admits the usual representation:

$$u(\lambda) = \alpha + \beta\lambda + \int \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1 + \xi^2} \right) d\sigma(\xi) \tag{4.9}$$

Now, as $\lambda \to \infty$, we have

$$w(\lambda) = \log \frac{1}{-\lambda} + \frac{1}{2\lambda^2} \int \xi^2 \, d\mathbf{n}(\xi) + \cdots$$

and consequently

$$\begin{aligned} u(\lambda) &= -\frac{1}{\lambda} \exp \left[\frac{1}{2\lambda^2} \int \xi^2 \, d\mathbf{n}(\xi) + \cdots \right] \\ &= -\frac{1}{\lambda} \left[1 + \frac{1}{2\lambda^2} \int \xi^2 \, d\mathbf{n}(\xi) + \cdots \right] \end{aligned}$$

Identifying with (4.9), one gets

$$u(\lambda) = \int \frac{1}{\xi - \lambda} \, d\sigma(\xi)$$

and

$$\int d\sigma(\xi) = 1, \qquad \int \xi \, d\sigma(\xi) = 0 \qquad \int \xi^2 \, d\sigma(\xi) = \frac{1}{2} \int \xi^2 \, d\mathbf{n}(\xi), \ldots$$

which completes the proof when $\mathbf{n}$ has compact support. In the general case, one approximates $\mathbf{n}$ by probability measures $\mathbf{n}_k$ having compact supports and one chooses σ as a limit point of the sequence $\{\sigma_k; k \geqslant 1\}$ of corresponding probability measure whose existence has just been shown. The rest is plain. ∎

Remark. The above argument shows that we have the equality

$$\int \xi^2 \, d\sigma(\xi) = \frac{1}{2} \int \xi^2 \, d\mathbf{n}(\xi) \tag{4.10}$$

whenever $\int \xi^4 \, d\mathbf{n}(\xi) < +\infty$.

Lemma 4.5. Let u_1 and u_2 be conformal maps from $\mathbb{C}_+$ into $\mathbb{C}_+$ of the form (4.8) for probability measures σ_1 and σ_2 on $\mathbb{R}$ satisfying

$$\int \xi \, d\sigma_1(\xi) = \int \xi \, d\sigma_2(\xi) = 0$$

$$\int \xi^2 \, d\sigma_1(\xi) < +\infty, \qquad \int \xi^2 \, d\sigma_2(\xi) < +\infty$$

Then we have

$$\int \xi^2 \, d\sigma_2(\xi) \leqslant \int \xi^2 \, d\sigma_1(\xi)$$

whenever

$$u_1(\mathbb{C}_+) \subset u_2(\mathbb{C}_+)$$

Proof. Set $u(\lambda) = u_2^{-1}(u_1(\lambda))$ for $\lambda \in \mathbb{C}_+$. Then u maps conformally $\mathbb{C}_+$ into $\mathbb{C}_+$ and admits the usual representation:

$$u(\lambda) = \alpha + \beta\lambda + \int \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) d\sigma(\xi) \tag{4.11}$$

Now, our assumptions on σ_1 and σ_2 imply that

$$u_1(\lambda) = \frac{1}{-\lambda} \left[1 + \frac{1}{\lambda^2} \int \xi^2 \, d\sigma_1(\xi) + O\left(\frac{1}{|\lambda|^2} \right) \right]$$

and

$$u_2(\lambda) = \frac{1}{-\lambda} \left[1 + \frac{1}{\lambda^2} \int \xi^2 \, d\sigma_2(\xi) + O\left(\frac{1}{|\lambda|^2} \right) \right]$$

as $\lambda \to \infty$ while keeping $\operatorname{Im} \lambda \geqslant c > 0$. This implies that

$$u(\lambda) = \lambda + \frac{1}{\lambda} \left[\int \xi^2 \, d\sigma_1(\xi) \right] + O\left(\frac{1}{|\lambda|} \right)$$

in the same regime. Comparing with (4.11), one obtains

$$u(\lambda) = \lambda + \int \frac{1}{\xi - \lambda} \, d\sigma(\xi)$$

with

$$0 \leqslant \int d\sigma(\xi) = \int \xi^2 \, d\sigma_1(\xi) - \int \xi^2 \, d\sigma_2(\xi) \quad \blacksquare$$

Proof of Proposition 4.3. The functions w_1 and w_2 map conformally $\mathbb{C}_+$ into $\{z \in \mathbb{C}_+ ; 0 \leqslant \operatorname{Im} z \leqslant \pi\}$. Indeed, $\operatorname{Im} w_1'(\lambda) > 0$ and $\operatorname{Im} w_2'(\lambda) > 0$. Consequently, the functions $u_1(\lambda) = e^{w_1(\lambda)}$ and $u_2(\lambda) = e^{w_2(\lambda)}$ map conformally $\mathbb{C}_+$ into $\mathbb{C}_+$ and the above two lemmas give

$$\frac{1}{2} \int \xi^2 \, d\mathbf{n}_1(\xi) \geqslant \int \xi^2 \, d\sigma_1(\xi) \geqslant \int \xi^2 \, d\sigma_2(\xi) = \frac{1}{2} \int \xi^2 \, d\mathbf{n}_2(\xi)$$

where we have used the compactness of the support of $\mathbf{n}_2$ to derive the last equality. ∎

Remark. It is not clear if one can avoid the assumption (c). Indeed, we could not find a way to approximate the measures $\mathbf{n}_1$ and $\mathbf{n}_2$ by probability measures with compact support in such a way that the inclusion (4.6) is satisfied at each step of the approximation and that the second moments converge.

Our reconstruction procedure relies on the approximation of general w-functions satisfying (i)–(iii) from the beginning of this section by w-functions corresponding to periodic sequences. Their general form was discussed at the end of the last section and a reconstruction theorem exists for them. It was proven by Kac and van Moerbeke.[5,6] Lemma 4.7 below recasts this result in our framework of w-functions.

Let w be a general w-function satisfying properties (i)–(iii). Its range $w(\mathbb{C}_+)$ is given in Fig. 2. Let $c \geqslant \sup_{\lambda \in \mathbb{C}_+} \operatorname{Re} w(\lambda)$ and for each integer $N \geqslant 1$ we define the domain D_N by

$$D_N = (-\infty, c) \times (0, \pi) \Big\backslash \bigcup_{j=1}^{N-1} \{x + ij\pi/N;\ c_j \leqslant x \leqslant c\}$$

where $c_j = \sup\{x \in \mathbb{R};\ x + ij\pi/N \in w(\mathbb{C}_+)\}$. Recall Fig. 1.

We have the following result:

Lemma 4.6. (Kac–van Moerbeke). If the domain D_N is as above, there exists a sequence $\{(a_n, b_n);\ n \in \mathbb{Z}\}$ with period N, such that $a_n^{(N)} > 0$

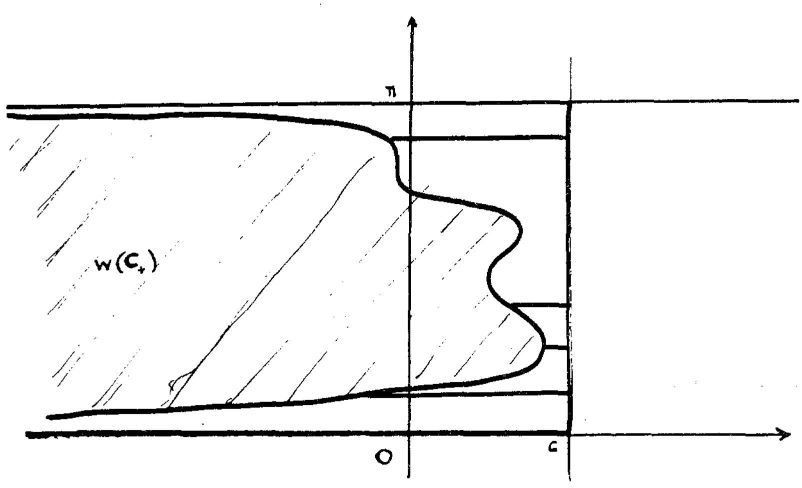

Figure 2

for all n, $c=-(1/N)\log a_1^{(N)}\cdots a_N^{(N)}$, and for which the corresponding w-function, say w_N, satisfies

$$w_N(\mathbb{C}_+)=D_N \tag{4.12}$$

Proof. The Riemann mapping theorem gives the existence of a conformal map, say w_N, from $\mathbb{C}_+$ onto D_N such that

$$w_N(\lambda)\sim\log\frac{1}{-\lambda}$$

as $\lambda\to\infty$ in $\mathbb{C}_+$. When λ varies through $\mathbb{R}$ from $-\infty$ to $+\infty$, $w_N(\lambda)$ varies through the boundary of D_N, and setting

$$\Delta(\lambda)=2\operatorname{ch}[N(w_N(\lambda)-c)]$$

one gets that:

1. $\operatorname{Im} w_N(\lambda)=0$ and so $\Delta(\lambda)\geqslant 2$ for $\lambda\in(-\infty,\lambda_1^+]$ for some $\lambda_1^+\in\mathbb{R}$; then, $\operatorname{Re} w_N(\lambda)=c$ and $\operatorname{Im} w_N(\lambda)$ increases from 0 to π/N, so that $\Delta(\lambda)=2\cos[N\operatorname{Im} w_N(\lambda)]$ decreases from 2 to -2 when λ increases from λ_1^+ to some $\lambda_1^-\in\mathbb{R}$
2. Then, $\operatorname{Re} w_N(\lambda)$ goes from c to c_1 while $\operatorname{Im} w_N(\lambda)$ remains constant, equal to π/N, so that $\Delta(\lambda)$ goes from -2 to something possibly smaller and back to -2 when λ increases from λ_1^- to some λ_2^-.
3. Then $\operatorname{Re} w_N(\lambda)=c$ while $\operatorname{Im} w_N(\lambda)$ increases from π/N to $2\pi/N$, so that $\Delta(\lambda)$ increases from -2 to 2 when λ increases from λ_2^- to some $\lambda_2^+\in\mathbb{R}$

Finally, we obtain $\lambda_1^+,\dots,\lambda_N^+$ roots of $\Delta(\lambda)=-2$ and $\lambda_1^-,\dots,\lambda_N^-$ roots of $\Delta(\lambda)=-2$, with the result shown in Fig. 3. This figure corresponds to the case N even. When N is odd, the last band is $[\lambda_N^+,\lambda_N^-]$ and $\Delta(\lambda)\leqslant -2$ for $\lambda\geqslant\lambda_N^-$. Note that $\Delta(\lambda)$ is real for λ real, so that $\Delta(\lambda)=\overline{\Delta(\bar\lambda)}$ defines an analytic continuation to the lower half-plane. Now $\Delta(\lambda)$ is an entire function. Actually $\Delta(\lambda)$ is a polynomial of degree N beause $\Delta(\lambda)\sim\lambda^N$ for $\lambda\to\infty$, $\lambda\in\mathbb{C}_+$.

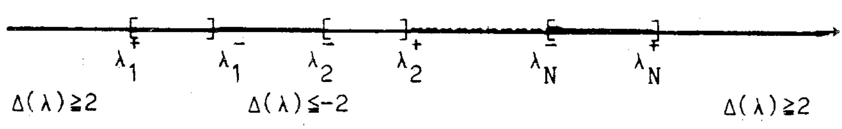

Figure 3

Now, we can choose numbers $\mu_1, \ldots, \mu_{N-1}$ such that μ_j belongs to the jth gap and impose the constraint $a_1^{(N)} \cdots a_N^{(N)} = e^{-Nc}$. Under these conditions we can use the Kac–van Moerbeke reconstruction theory[5,6] to finish the proof. ∎

The final technical lemma is the following approximation result. Its idea (and proof) is very similar to the Caratheodory convergence theorem. See, for example, Ref. 4.

Lemma 4.7. Let w be a Herglotz function satisfying the properties (i)–(iii) of (4.1) and let us assume that the distribution function $\mathbf{n}(\xi)$ of the probability measure $\mathbf{n}$ giving the representation (4.2) is smooth and satisfies

$$\int \xi^2 \, d\mathbf{n}(\xi) < +\infty, \qquad \int \xi \, d\mathbf{n}(\xi) = 0 \tag{4.13}$$

Then, the Herglotz functions w_N defined in Lemma 4.6 converge to w compact uniformly on $\mathbb{C}_+$.

Proof. By Proposition 4.3 we have

$$\int \xi^2 \, d\mathbf{n}_N(\xi) \leqslant \int \xi^2 \, d\mathbf{n}(\xi) \tag{4.14}$$

for all $N \geqslant 1$, so that our assumption (4.13) implies the existence of an increasing sequence $\{N_k; k \geqslant 1\}$ of integers and of a probability measure $\tilde{\mathbf{n}}$ on $\mathbb{R}$ that satisfies the condition (4.13) and such that

$$\lim_{k \to \infty} n_{N_k} = \tilde{\mathbf{n}}$$

in the sense of the weak convergence of measures. Also, the uniform bound implies that the function $\tilde{w}$ defined by

$$\tilde{w}(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\tilde{\mathbf{n}}(\xi)$$

is also a conformal mapping from $\mathbb{C}_+$ into $\mathbb{C}_+$ [because Im $\tilde{w}'(\lambda) > 0$] and is the compact uniform limit of the functions w_{N_k} on $\mathbb{C}_+$. We show that $\tilde{w} = w$.

Let us first assume that the distribution function of the probability measure $\mathbf{n}$ is smooth. This assumption implies the smoothness of the boundary of $w(\mathbb{C}_+)$ and the proof of the Caratheodory convergence theorem shows that we actually have $w(\mathbb{C}_+) = \tilde{w}(\mathbb{C}_+)$, so that w and $\tilde{w}$ are conformal mappings from $\mathbb{C}_+$ into $\mathbb{C}_+$ with the same range. Hence, the function

$u(\lambda)$ defined by $u(\lambda) = \tilde{w}^{-1}(w(\lambda))$ maps $\mathbb{C}_+$ onto $\mathbb{C}_+$ conformally and is consequently of the form

$$u(\lambda) = (a\lambda + b)/(c\lambda + d)$$

for some real numbers a, b, c, and d satisfying $ad - bc > 0$. Thus

$$w(\lambda) = \tilde{w}\left(\frac{a\lambda + b}{c\lambda + d}\right)$$

Now, the limit $\tilde{w}(\xi + i0)$ exists and is finite for any $\xi \in \mathbb{R}$ (see the Appendix of Ref. 3). Using this fact for $\xi = a/c$ and the fact that

$$w(iy) \sim \log(i/y)$$

as $y \to +\infty$, we obtain $c = 0$, $a \neq 0$ and we can pick $a = 1$. Finally, the condition $\int \xi \, d\mathbf{n}(\xi) = \int \xi \, d\mathbf{n}(\xi) = 0$ implies $b/d = 0$, which proves $w = \tilde{w}$.

In the general case, we first approximate the probability measure $\mathbf{n}$ by probability measures $\mathbf{n}_\varepsilon$ with smooth distribution functions (choose for example

$$d\mathbf{n}_\varepsilon(\xi) = \int j_\varepsilon(\xi - \eta) \, d\mathbf{n}(\eta)$$

for a smooth approximate identity $\{j_\varepsilon; \varepsilon > 0\}$ on $\mathbb{R}$), the corresponding w-functions

$$w_\varepsilon(\lambda) = \int \log \frac{1}{\xi - \lambda} \, d\mathbf{n}_\varepsilon(\xi)$$

being compact uniformly convergent to the w-function from which we are starting. Then one can use the above argument ∎

The main result of the paper is the following:

Theorem 4.8. Let w be a Herglotz function satisfying properties (i)–(iii) of (4.1). Then there exists a stationary sequence $\{(a_n, b_n); n \in \mathbb{Z}\}$ such that

$$\mathbf{E}\{a_0^2 + |\log a_0| + b_0^2\} < +\infty \tag{4.15}$$

and such that

$$w(\lambda) = \mathbf{E}\{\log m_+(\lambda)\}, \qquad c \geqslant -\mathbf{E}\{\log a_0\} \tag{4.16}$$

Proof. Let us construct the domains D_N from $w(\mathbb{C}_+)$ as described in Fig. 2, and for each integer N we can redefine the periodic sequence

$\{(a_n^{(N)}, b_n^{(N)});\ n\in\mathbb{Z}\}$ given in Lemma 4.7 as a stationary process given by the coordinate projections, say (a_n, b_n), on the product space $\Omega=[(0,\infty)\times\mathbb{R}]^{\mathbb{Z}}$ for some probability measure $\mathbf{P}_N$. The corresponding w-function

$$w_N(\lambda)=\mathbf{E}_{\mathbf{P}_N}\{\log m_+(\lambda)\}$$

satisfies $w_N(\mathbb{C}_+)=D_N$. Notice that

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}_N}\{2a_0^2+b_0^2\} &= \int \xi^2\, dn_N(\xi) \\
&\qquad \text{(because of Lemma 3.1)} \\
&\leqslant \int \xi^2\, dn(\xi) \\
&\qquad \text{(because of Lemma 4.3)} \\
&< +\infty
\end{aligned} \tag{4.17}$$

This implies the tightness of the family $\{\mathbf{P}_N;\ N\geqslant 1\}$ of probability measures on Ω. Let $\mathbf{P}$ be any limit point of this family (for notational convenience we will consider that $\mathbf{P}$ is actually the limit of the whole sequence $\{\mathbf{P}_N;\ N\geqslant 1\}$). $\mathbf{P}$ is invariant for the shift operator on Ω, so that the sequence $\{(a_n, b_n);\ n\in\mathbb{Z}\}$ is still stationary. Moreover,

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}}\{2a_0^2+b_0^2\} &\leqslant \liminf_{N\to\infty} \mathbf{E}_{\mathbf{P}_N}\{2a_0^2+b_0^2\} \\
&< +\infty
\end{aligned}$$

because of (4.17). If $c\geqslant \sup_{\lambda\in\mathbb{C}_+} \operatorname{Re} w(\lambda)$, our construction gives

$$c=\mathbf{E}_{\mathbf{P}_N}\left\{\log\frac{1}{a_0}\right\}=\mathbf{E}_{\mathbf{P}_N}\left\{\log^+\frac{1}{a_0}\right\}-\mathbf{E}_{\mathbf{P}_N}\{\log^+ a_0\} \tag{4.18}$$

Also, (4.17) implies that

$$\lim_{N\to\infty} \mathbf{E}_{\mathbf{P}_N}\{\log^+ a_0\}=\mathbf{E}_{\mathbf{P}}\{\log^+ a_0\}< +\infty \tag{4.19}$$

Moreover,

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}}\left\{\log^+\frac{1}{a_0}\right\} &\leqslant \liminf_{N\to\infty} \mathbf{E}_{\mathbf{P}_N}\left\{\log^+\frac{1}{a_0}\right\} \\
&= c+\mathbf{E}_{\mathbf{P}}\{\log^+ a_0\}< +\infty
\end{aligned} \tag{4.20}$$

because of (4.18) and (4.19). Indeed, Skorokhod's representation theorem[14] allows for an interpretation in terms of almost sure convergence and one can apply Fatou's Lemma. Relations (4.19) and (4.20) imply

$$\mathbf{E}_{\mathbf{P}}\{|\log a_0|\} < +\infty$$

which completes the proof of (4.15). Using again Skorokhod's theorem, one can think of $(a_n^{(N)}, b_n^{(N)})$, once redefined on the same probability space, as converging almost surely to the $\{(a_n, b_n); n \in \mathbb{Z}\}$. Consequently, our discussion in Section 2 leading to (2.12) gives the almost sure convergence of the corresponding m-functions and the deterministic bounds (2.8) give

$$\lim_{N \to \infty} \mathbf{E}_{\mathbf{P}_N}\{\log m_+(\lambda)\} = \mathbf{E}_{\mathbf{P}}\{\log m_+(\lambda)\}$$

by Lebesgue's dominated convergence theorem. Together with Lemma 4.7, this gives (4.16) and the proof is now complete. ∎

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